

Coloration of K_7^- -minor free graphs¹

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Abstract

Hadwiger's conjecture says that every K_t -minor free graph is $(t - 1)$ -colorable. This problem has been proved for $t \leq 6$ but remains open for $t \geq 7$. K_7 -minor free graphs have been proved to be 8-colorable (Albar & Gonçalves, 2013). We prove here that K_7^- -minor free graphs are 7-colorable, where K_7^- is the graph obtained from K_7 by removing one edge.

1 Introduction

A minor of a graph G is a graph obtained from G by a succession of edge deletions, edge contractions and vertex deletions. All graphs we consider are simple, i.e. without loops or multiple edges.

Hadwiger's conjecture says that every t -chromatic graph G (i.e. $\chi(G) = t$) contains K_t as a minor. This conjecture has been proved for $t \leq 6$, where the case $t = 5$ is equivalent to the Four Color Theorem by Wagner's structure theorem of K_5 -minor free graphs, and the case $t = 6$ has been proved by Robertson, Seymour and Thomas [7]. The conjecture remains open for $t \geq 7$.

In [1], the author and D. Gonçalves proved that K_7 -minor graphs are 8-colorable.

In [5], Kawarabayashi and Toft proved that any K_7 and $K_{4,4}$ -minor free graph is 6-colorable by using the fact that a $K_{4,4}$ -minor free graph contains at most $4n - 8$ edges. In particular, this implies that it contains some vertices of degree 7. In their proof they show that most of these vertices in a 7-chromatic critical graph (i.e. such that every strict minor of this graph is 6-colorable) are

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contained in a K_5 subgraph and use these subgraphs and the 7-connectivity of a 7-chromatic critical graph to find a K_7 -minor.

We use here similar techniques to prove the following theorem.

Theorem 1 *Every K_7^- -minor free graph is 7-colorable.*

2 Proof of Theorem 1

Let G be a minimal counter example to Theorem 1, i.e. a minimal K_7^- -minor free 8-chromatic critical graph,

First we will prove that a lot of vertices of degree 8 are contained in K_5 subgraphs and then we will apply some techniques introduced in [5] to conclude.

We will use the following theorem of Jakobsen to prove that K_7^- -minor free graphs are 8-degenerate.

Theorem 2 (Jakobsen, 1983, [4]) *Every graph with at least 7 vertices and at least $\frac{9}{2}n - 12$ edges has a K_7^- -minor or is a $(K_{2,2,2,2}, K_6, 4)$ -cockade.*

We also need the following theorem of Mader.

Theorem 3 (Mader, 1968, [6]) *Any k -chromatic critical graph that is not isomorphic to K_7 is 7-connected for $k \geq 7$.*

Hence G is 7-connected, and thus is not a $(K_{2,2,2,2}, K_6, 4)$ -cockade. Thus we can deduce the following corollary of these two theorems.

Corollary 4 *G has less than $\frac{9}{2}n - 12$ edges.*

We also need the following folklore lemma (see [1] for a proof).

Lemma 5 (Folklore) *In a 8-chromatic critical graph G , G has minimum degree at least 7 and for any vertex u of degree 7 (resp. 8), then the graph induced by $N(u)$ has no stable of size 2 (resp. 3).*

In particular, this lemma implies that G has minimum degree at least 8 because if G contains a vertex u of degree 7 then $N(u)$ has no stable set of size 2 and thus G contains a K_7 -minor, a contradiction. We will use vertices of degree 8 and their neighborhoods to find a K_7^- -minor. The following lemma ensures the existence of such vertices.

Lemma 6 *G has at least 25 vertices of degree 8.*

Proof. By Corollary 4, G has less than $\frac{9}{2}n - 12$ edges. Suppose that G has at most 24 vertices of degree 8. By Lemma 5, G has no vertices of degree strictly less than 8, so we have that :

$$|E(G)| \geq \frac{9(n - 24) + 8 * 24}{2} = \frac{9}{2}n - 12,$$

a contradiction. □

Lemma 7 *Let u be a vertex of degree 8, then either $N(u)$ contains K_4 as a subgraph or $N(u)$ contains the graph $C_8^{1,2}$, i.e. the circulant graph on 8 vertices with jumps 1, 2 (see Figure 1), as a subgraph.*

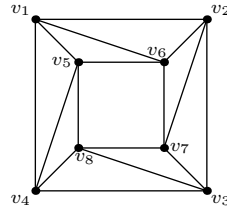


Fig. 1. The graph $C_8^{1,2}$

Before proving Lemma 7, let us introduce some material. The following lemma can be immediatly deduced from the four-color theorem.

Lemma 8 *Let x, y and z be three vertices of G , then $G - \{x, y, z\}$ is 4-connected and non-planar.*

Proof. The first part of the lemma is obvious by the 7-connectivity of G . Suppose now that there exists $x, y, z \in V(G)$ such that $G - \{x, y, z\}$ is planar. By the Four Color Theorem, if $G - \{x, y, z\}$ is planar then it is 4-colorable, thus G is 7-colorable, a contradiction. □

We need the following definition and theorem introduced by Robertson, Seymour and Thomas in [7].

Definition 9 *Let H be a graph and $T = \{v_1, v_2, v_3\}$ be a triangle. H is said triangular with respect to T if one of the following holds.*

- *For some i ($1 \leq i \leq 3$), $H \setminus \{v_i\}$ has maximum valency at most 2, and either $H \setminus \{v_i\}$ is a circuit or it has no circuit.*
- *All vertices of H have valency at most 3, there is at most one 3-valent vertex $v \neq v_1, v_2, v_3$, and $H \setminus \{v_1, v_2, v_3\}$ has no circuit.*
- *All vertices of H have valency at most 3, there is a triangle C with $v_1, v_2, v_3 \notin V(C)$, every 3-valent vertex of H is in $\{v_1, v_2, v_3\} \cup V(C)$, and every circuit of H except these two triangles meets both $\{v_1, v_2, v_3\}$ and $V(C)$.*

Theorem 10 (Robertson, Seymour & Thomas, 1993, [7]) *Let v_1, v_2, v_3 be a triangle T in a 4-connected non-planar graph H . Let Z be an induced subgraph of H such that $v_1, v_2, v_3 \in Z$ and Z is not triangular with respect to T . Then H has a K_5 -minor $(v_1, v_2, v_3, Z_1, Z_2)$ in H such that $Z_1 \cap Z, Z_2 \cap Z \neq \emptyset$.*

Let us now prove Lemma 7.

Proof. Let u be a vertex of degree 8 of G and suppose that the graph induced by $N(u)$ is K_4 -free.

Claim 11 $N(u)$ is 4-connected.

Proof. Let (A, B) be a minimal separation of $N(u)$. Since there is no stable of size 3 in $N(u)$ by Lemma 5, for each pair of vertices of $v, v' \in A \setminus B$ and any vertex in $w \in B \setminus A$, $\{v, v', w\}$ contain at least one edge. This edge cannot be vw or $v'w$ because (A, B) is a separation of $N(u)$. So this must be the edge vv' . We deduce that both $A \setminus B$ and $B \setminus A$ are complete graphs.

If (A, B) is a separation of order 1, then either $|A \setminus B| \geq 4$ or $|B \setminus A| \geq 4$. By the previous remark, $N(u)$ contains a K_4 subgraph, a contradiction.

Let suppose that (A, B) is a separation of order 2, then in this case $|A \setminus B| = |B \setminus A| = 3$. Let $v \in A \cap B$. Since the graph induced by $N(u)$ is K_4 -free and since $A \setminus B$ and $B \setminus A$ are triangles, there is one vertex $w \in A \setminus B$ such that vw is not an edge. In the same way, there is a vertex $w' \in B \setminus A$ such that vw' is not an edge. Since (A, B) is a separation of $N(u)$, then $\{v, w, w'\}$ is a stable set of size 3, a contradiction.

Let now suppose that (A, B) is a separation of order 3. By the previous remark, $|B \setminus A| \leq 3$ and $|A \setminus B| \leq 3$. Since $|N(u)| = 8$ and $|A \cap B| = 3$, we can assume without loss of generality that $|A \setminus B| = 3$ and $|B \setminus A| = 2$. Let $A \cap B = \{s_1, s_2, s_3\}$ and let $B \setminus A = \{b_1, b_2\}$. Suppose that there is a vertex s_i , $1 \leq i \leq 3$ and a vertex b_j , $1 \leq j \leq 2$, such that $s_i b_j$ is not an edge, then since $N(u)$ has no stable set of size 3, s_i is adjacent to all the vertices of the triangle $A \setminus B$ but then $N(u)$ contains a K_4 -subgraph, a contradiction. Thus we can assume that b_1 and b_2 are adjacent to all the vertices of $A \cap B$.

Now since $N(u)$ is K_4 -free, $A \cap B$ is a stable set because if say $s_i s_j$ are adjacent for $1 \leq i < j \leq 3$ then $\{b_1, b_2, s_i, s_j\}$ would be a K_4 -subgraph, a contradiction. But then $A \cap B$ is a stable set of size 3, a contradiction. \square

Claim 12 $N(u)$ is planar.

Proof. Assume that $N(u)$ is non-planar. Since $N(u)$ is 4-connected by Claim 11,

then $N(u)$ contains a K_5 -minor by Wagner's theorem [9]. Since G is not isomorphic to $N[u] = \{u\} \cup N(u)$ as it contains at least 25 vertices, then we can find $w \in G \setminus N[u]$. By the 7-connectivity of G , there is 7-vertex disjoint paths between w and u . Let denote them by P_1, P_2, \dots, P_7 . We can always assume that these paths are minimal in length and thus that these paths intersect $N(u)$ in at most one vertex. If there is 8 vertex-disjoint paths between u and w , then there exists 8 vertex-disjoint paths between w and every vertex of $N(u)$. Since $N(u)$ contains a K_5 -minor, then $N(u)$, together with u , w and the 8 paths between $N(u)$ and w , contains a K_7^- -minor, a contradiction.

So now, let v be the only vertex of $N(u)$ which is not contained in any of the 7 paths between w and $N(u)$. By Ramsey's theorem, since $N(u) \setminus \{v\}$ has 7 vertices and no stable set of size 3, then it contains a triangle. Denote by v_1, v_2 and v_3 its vertices. Since $N(u)$ is 4-connected, $N(u)$ is not triangular with respect to $\{v_1, v_2, v_3\}$, and since it is 4-connected and non-planar, then by Theorem 10, there exists Z_1 and Z_2 such that $(v_1, v_2, v_3, Z_1, Z_2)$ is a K_5 -minor. Since $N(u)$ does not contain any K_4 subgraphs, then $|Z_1|, |Z_2| \geq 2$, so both sets Z_1 and Z_2 intersect at least one of the 7 paths P_i , $1 \leq i \leq 7$. Thus $(v_1, v_2, v_3, Z_1, Z_2, u, \bigcup_{1 \leq i \leq 7} (V(P_i) \setminus N[u]))$ is a K_7^- -minor, a contradiction. \square

Claim 13 $N(u)$ does not contain any vertex of degree 6 or greater in $G[N(u)]$.

Proof. Suppose that $N(u)$ contains a vertex v of degree greater or equal than 6, then the graph induced by $N(v)$ in $N(u)$ contains no stable set of size 3, but then by Ramsey's theorem it contains a triangle. Thus $N(u)$ contains a K_4 -subgraph, a contradiction. \square

Claim 14 The neighborhood of any vertex of $N(u)$ is a 4-path, a 4-cycle or a 5-cycle.

Proof. Let v be a vertex of degree 4 in $N(u)$, denote by v_1, v_2, v_3 and v_4 its neighbors. Suppose that its neighborhood is not a path nor a 4-cycle. Since the neighborhood of v is triangle-free and does not contain a stable set of size 3, it must be two disjoint edges, say v_1v_2 and v_3v_4 . Denote by x, y and z , the three vertices in $N(u) \setminus \{v, v_1, v_2, v_3, v_4\}$. $\{x, y, z\}$ is a triangle because otherwise there is stable set of size 3 with v .

Every vertex in $\{v_1, v_2, v_3, v_4\}$ sees exactly two vertices in $\{x, y, z\}$ because, either there would be a vertex of degree at most 3 in $N(u)$, contradicting the 4-connectivity of $N(u)$, or if one of these vertices is adjacent to the three vertices x, y and z then $N(u)$ would contain a K_4 subgraph, another contradiction. Then as there are 8 edges between $\{v_1, \dots, v_4\}$ and $\{x, y, z\}$, there exists one vertex of degree 5 in $\{x, y, z\}$ say x . By symmetry, we can assume that x is adjacent to v_1, v_2 and v_3 . Now since every vertex in $\{v_1, v_2, v_3, v_4\}$ is adjacent

to two vertices in $\{x, y, z\}$, v_1 , v_2 and v_3 are adjacent to either y or z . But then $(v_1, v_2, v_3, v, x, \{y, z\})$ is a $K_{3,3}$ -minor, contradicting Claim 12.

If v is a vertex of degree 5, then since $N(v)$ does not contain any stable set of size 3 and any triangle, then $N(v)$ can only be isomorphic to the cycle of length 5. \square

Since $N(u)$ is planar, it has at most 18 edges by Mader's theorem, so it contains at least one vertex of degree 4. Let v be such a vertex. Denote by v_1, v_2, v_3 and v_4 its neighbors and x, y and z its 3 non-neighbors. Then $C = \{v_1, v_2, v_3, v_4\}$ can induce either a 4-path or a 4-cycle.

Suppose that C is a 4-path. Now the neighborhood of v_1 cannot induce a 4-cycle or a 5-cycle because this would contradict that v has degree 4. So v_1 has degree 4 and its neighborhood is a 4-path. By symmetry we can assume that v_1 's neighbors are the 4-path vv_2xy . Moreover $\{z, v_3, v_4\}$ is a triangle since otherwise $N(u)$ would contain a stable set of size 3 with v_1 . Now y is adjacent to at least 1 other vertex in C because it would be of degree 3 otherwise, contradicting the 4-connectivity of $N(u)$. Planarity forces y to be adjacent to v_4 , but then $N(u)$ contains $C_8^{1,2}$.

Now suppose that C is the 4-cycle $v_1v_2v_3v_4$. Suppose that v_1 has degree 4 and assume that v_1 's neighborhood is a 4-path, say v_4vv_2x . Now $\{v_3, y, z\}$ is also a triangle because otherwise there is a stable set of size 3 with v_1 . As y and z have degree at least 4 in $N(u)$ and as $y, z \notin N(v) \cup N(v_1)$ then y and z are both adjacent to at least one vertex in $\{v_2, v_4\}$. Moreover y and z cannot be both adjacent to the same vertex because otherwise there would be a K_4 -subgraph with v_3 . So either y is adjacent to v_2 and the z is adjacent to v_4 either z is adjacent to v_2 and the y is adjacent to v_4 . In both cases, after removing the edge v_2v_3 , the graph is isomorphic to $C_8^{1,2}$. Note that the same argument applies when v_1 's neighborhood is a 4-cycle by also removing the edge v_4x at the end.

Suppose now that v_1 has degree 5 so we can assume that its neighborhood is the 5-cycle v_4vv_2xy . Then z is adjacent to v_3 because otherwise $\{v_1, v_3, z\}$ is a stable set of size 3. Since z has degree at least 4 in $N(u)$ it is also adjacent to at least one vertex in the set $\{v_2, v_4\}$. But if z is adjacent to v_2 then after removing the edge v_1v_2 , the graph is isomorphic to $C_8^{1,2}$, and if z is adjacent to v_4 then after removing the edge v_1v_4 , the graph is isomorphic to $C_8^{1,2}$. \square

Lemma 15 *Let u and u' be two degree 8 vertices of G such that $N(u)$ and $N(u')$ contain the graph $C_8^{1,2}$ as a subgraph, then u and u' are not adjacent.*

Proof. Let suppose that u and u' are adjacent. Since every vertex of $N(u)$

has degree at least 4 and $u' \in N(u)$ by hypothesis, denote by v_1, v_2, v_3 and v_4 the four neighbors of u' in the subgraph $C_8^{1,2}$ of $N(u)$. These four vertices induce a path in this subgraph, say $v_1v_2v_3v_4$. Let denote by w_1, w_2 and w_3 the vertices of $N(u) \setminus \{u', v_1, v_2, v_3, v_4\}$ in a way that w_1 is the only vertex adjacent to both v_1 and v_2 and w_2 is the one adjacent to v_3 and v_4 . Now consider $H = G \setminus \{u, w_1, w_2\}$. H is 4-connected and non-planar by Lemma 8. Let $Z = \{w_3, u', v_1, v_2, v_3, v_4\}$, then Z is not triangular with respect to $\{u', v_1, v_2\}$ because u' has degree 4 in Z . Thus by Theorem 10, there exists Z_1 and Z_2 such that (u', v_1, v_2, Z_1, Z_2) is a K_5 -minor in H and such that $Z_1 \cap Z, Z_2 \cap Z \neq \emptyset$. But then $(u, u', v_1, v_2, Z_1, Z_2, \{w_1, w_2\})$ is a K_7^- -minor in G , a contradiction. \square

The following lemma is the key to prove that a lot of degree 8 vertices are contained in a K_5 .

Lemma 16 *Let u and u' be two vertices of degree 8 such that $N(u)$ and $N(u')$ contain the graph $C_8^{1,2}$ as a subgraph and $|N(u) \cup N(u')| \geq 9$, then G contains a K_7^- -minor.*

Proof. By Lemma 15, we can assume that u and u' are not adjacent. Denote by v_1, \dots, v_8 the vertices of $N(u)$ as shown in Figure 1. Since G is 7-connected, there is at least 7 internally disjoint paths between u and u' that induce 7 disjoint paths between $N(u)$ and $N(u')$. Note that theses paths can be of length 0 if the two neighborhoods intersect. By contracting the non-zero length paths, we obtain a graph with $|N(u) \cup N(u')| = 9$. From now on, we consider only this new graph G' . By construction of G' , $N(u)$ still contain a $C_8^{1,2}$ -subgraph.

By symmetry of $C_8^{1,2}$, we can assume that v_1 is the only neighbor of u which is not a neighbor of u' . In particular, we have that $v_i \in N(u')$ for all $i \geq 2$. But then $(u, \{u', v_5\}, v_2, v_3, v_6, v_7, \{v_1, v_4, v_8\})$ is a K_7^- -minor (only v_3 and v_6 are not adjacent) of G' and thus a K_7^- -minor of G , a contradiction. \square

Claim 17 *Let u and u' be two vertices of degree 8 such that $N(u)$ and $N(u')$ contain the graph $C_8^{1,2}$ as a subgraph, then $N(u) \neq N(u')$.*

Proof. Suppose that there exists two vertices u and u' of degree 8 such that $N(u) = N(u')$. Then we can create a K_7^- -minor in G by using the same argument as in the proof of Lemma 16, a contradiction. \square

Claim 18 *At most one vertex of degree 8 have a neighborhood containing the graphs $C_8^{1,2}$ as a subgraph.*

Proof. Suppose that there exists two vertices of degree 8 such that their

neighborhood contains the graph $C_8^{1,2}$ as a subgraph. By Claim 17, these two vertices have a different neighborhood. By Lemma 16, this imply that there is a K_7^- -minor in G , a contradiction. \square

Lemma 19 *There is at least 5 different K_5 in G .*

Proof. By Lemma 6, there is at least 25 vertices of degree 8 and by Lemma 18, there is at most one vertices of degree 8 containing the graph $C_8^{1,2}$ as a subgraph of their neighborhood. By Lemma 7, this imply that there is at least 24 vertices of degree 8 that contains a K_4 in their neighbourhood. As every K_5 -subgraph can contain at most 5 vertices of degree 8, this finally imply that there is at least $\lceil \frac{24}{5} \rceil = 5$ different K_5 -subgraph in G . \square

The following lemma is the last key to the proof. It uses techniques introduced by Kawarabayashi and Toft [5].

Lemma 20 *There is 3 different copies of K_5 L_1 , L_2 and L_3 such that $|L_1 \cup L_2 \cup L_3| \geq 12$.*

Proof. Assume by contradiction that no three copies of K_5 , denoted L_i , L_j and L_k , are such that $|L_i \cup L_j \cup L_k| \geq 12$.

The next claim follows easily from the 7-connectivity of G .

Claim 21 *G does not contain a K_6^- subgraph.*

Proof. Suppose that G contains a K_6^- subgraph. Since G is not isomorphic to K_6^- , there exists a vertex that is not contained in this K_6^- subgraph. Since G is 7-connected, by Menger's theorem there are 7 vertex-disjoint paths between x and the vertices of the K_6^- subgraph. This induces a K_7^- -minor, a contradiction. \square

Claim 22 *Two different K_5 intersects on at most 2 vertices.*

Proof. Let L_1 and L_2 be two copies of K_5 of G and suppose that they intersect on 4 vertices, then G contains a K_6^- as a subgraph, contradicting Claim 21. If they intersect on 3 vertices, then denote by S the set of vertices in $L_1 \cap L_2$ and by H the set of vertices of $L_1 \Delta L_2$. By Lemma 8, $G \setminus S$ is 4-connected and non-planar so by (2.6) of [7] there is a K_4 -minor rooted in H and a K_7 -minor in G , a contradiction. \square

Claim 23 *No two K_5 are disjoint.*

Proof. Assume that L_1 and L_2 are two disjoint copies of K_5 . For any copy of K_5 L_3 , since two copies of K_5 cannot intersect on 4 vertices and since $|L_1 \cup L_2 \cup L_3| < 12$, $|L_3 \cap L_1| \geq 2$ and $|L_3 \cap L_2| \geq 2$. By Claim 22, $|L_3 \cap L_1| = 2$ and $|L_3 \cap L_2| = 2$. Let $L_3 \cap L_1 = \{a, b\}$ and $L_3 \cap L_2 = \{c, d\}$.

Now $G \setminus \{a, b, c, d\}$ is 3-connected so by Menger's theorem there are 3 vertex disjoint paths P_1 , P_2 and P_3 between $L_1 \setminus \{a, b\}$ and $L_2 \setminus \{c, d\}$ but then $(a, b, c, d, V(P_1), V(P_2), V(P_3))$ is a K_7 -minor, a contradiction. \square

Claim 24 *No two K_5 intersect on exactly one vertex.*

Proof. Assume that $L_1 \cap L_2 = \{x\}$. Let L_3 be a copy of K_5 different from L_1 and L_2 . By Claim 23, L_3 intersects both L_1 and L_2 .

Suppose that $x \in L_3$. Since $|L_1 \cup L_2 \cup L_3| < 12$, $|L_1 \cup L_3| = |L_2 \cup L_3| = 2$. Let $y \in (L_1 \cap L_3) \setminus \{x\}$. $G \setminus \{x, y\}$ is 5-connected and non-planar by Lemma 8. Let $Z = (L_1 \cup L_2 \cup L_3) \setminus \{x, y\}$. Denote $T = L_2 \setminus \{x, y\} = \{v_1, v_2, v_3\}$. Z is not triangular with respect to T , hence there exists Z_1, Z_2 such that $(v_1, v_2, v_3, Z_1, Z_2)$ is a K_5 -minor in $G \setminus \{x, y\}$ and such that $Z_1 \cap Z, Z_2 \cap Z \neq \emptyset$. Moreover we can assume without loss of generality that y is adjacent to Z_1 . Thus $(v_1, v_2, v_3, Z_1, Z_2, x, y)$ is a K_7^- -minor in G (only y and Z_2 may not be adjacent), a contradiction.

Suppose now that $x \notin L_3$. Since $|L_1 \cup L_2| = 9$ and $|L_1 \cap L_2 \cap L_3| < 12$, by Claim 22, we can assume that $|L_3 \cap L_1| = 2$. Let us denote $L_3 \cap L_1 = \{a, b\}$.

If $|L_3 \cap L_2| = 1$, let $\{c\} = L_3 \cap L_2$. Now $G \setminus \{a, b, c, x\}$ is 3-connected. So by Menger's theorem, there are 3 vertex disjoint paths P_1 , P_2 and P_3 , between $(L_1 \cup L_3) \setminus \{a, b, c, x\}$ and $L_2 \setminus \{c, x\}$. Hence $(a, b, c, x, V(P_1), V(P_2), V(P_3))$ is a K_7 -minor, a contradiction.

If $|L_3 \cap L_2| = 2$, let $\{c, d\} = L_3 \cap L_2$. $G \setminus \{a, b, c, d, x\}$ is 2-connected, so by Menger's theorem, there are 2 vertex disjoint paths P_1 and P_2 between $(L_1 \cup L_3) \setminus \{a, b, c, d, x\}$ and $L_2 \setminus \{c, d, x\}$. But $(a, b, c, d, x, V(P_1), V(P_2))$ is a K_7 -minor, a contradiction. \square

Claim 25 *No two K_5 intersect on exactly two vertices.*

Proof. Assume that $L_1 \cap L_2 = \{x, y\}$. Let L_3 be a K_5 different from L_1 and L_2 . By Claims 22, 23 and 24, L_3 intersects each L_1 and L_2 on two vertices.

Suppose that $L_1 \cap L_2 \cap L_3 = \emptyset$ and let $L_1 \cap L_3 = \{u, v\}$ and $L_2 \cap L_3 = \{z, t\}$. Then $\{u, v, x, y, z, t\}$ is a K_6 -subgraph, a contradiction with Claim 21.

Suppose that $L_1 \cap L_2 \cap L_3 = \{x\}$ and let $(L_1 \cap L_2) \setminus \{x\} = \{y\}$, $(L_1 \cap L_3) \setminus \{x\} =$

$\{z\}$, $(L_2 \cap L_3) \setminus \{x\} = \{t\}$. Now $G \setminus \{x, t\}$ is 5-connected and non-planar. Let $Z = (L_1 \cup L_2) \setminus \{x, t\}$ and let $T = \{v_1, v_2, v_3\} = L_2 \setminus \{x, t\}$. Z is not triangular with respect to T , so there exists Z_1 and Z_2 such that $(v_1, v_2, v_3, Z_1, Z_2)$ is a K_5 -minor in $G \setminus \{x, t\}$. Without loss of generality, we can assume that $z \in Z_1$ but then $(v_1, v_2, v_3, Z_1, Z_2, x, t)$ is a K_7^- -minor in G , a contradiction.

Finally, suppose that $L_1 \cap L_2 \cap L_3 = \{x, y\}$, $G \setminus \{x, y\}$ is 5-connected and non-planar. Let $Z = (L_1 \cup L_2 \cup L_3) \setminus \{x, y\}$ and $T = \{v_1, v_2, v_3\} = L_1 \setminus \{x, y\}$. Z is not triangular with respect to T so there exists Z_1 and Z_2 such that $(v_1, v_2, v_3, Z_1, Z_2)$ is a K_5 -minor in $G \setminus \{x, y\}$, but then $(v_1, v_2, v_3, Z_1, Z_2, x, y)$ is a K_7 -minor in G , a contradiction. \square

Claims 23, 24 and 25 together with Claim 22 conclude the proof of the lemma. \square

We conclude the proof of Theorem 1 by using the following theorem due to Kawarabayashi and Toft [5].

Theorem 26 (Kawarabayashi & Toft, 2005, [5]) *Let G be a 7-connected graph with at least 19 vertices. Suppose that G contains three K_5 , say L_1 , L_2 and L_3 , such that $|L_1 \cup L_2 \cup L_3| \geq 12$, then G contains a K_7 -minor.*

Applying this theorem to the three K_5 given by Lemma 20 gives us a contradiction.

3 Conclusion

We have seen that K_7^- -minor free graphs are 7-colorable. The techniques used here are not sufficient to prove that K_7 -minor free graphs are 7-colorable because we then have to deal with "sparse" neighborhoods of degree 8 and 9 vertices. However, since 6-connected K_8^- -minor free graphs are 10-degenerated [8], we wonder whether similar techniques can be extended to prove that K_8^- -minor free graphs are 9-colorable. Currently the best bound for K_8^- -minor free graphs is given by the fact that K_8 -minor free graphs are 10-colorable [1].

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